

Harmonic Superspaces from Superstrings

P. A. Grassi ^{a,b,1}, and P. van Nieuwenhuizen ^{a,2}

^(a) *C.N. Yang Institute for Theoretical Physics,
State University of New York at Stony Brook, NY 11794-3840, USA*

^(b) *Dipartimento di Scienze, Università del Piemonte Orientale,
C.so Borsalino 54, Alessandria, 15100, ITALY*

We derive harmonic superspaces for $N = 2, 3, 4$ SYM theory in four dimensions from superstring theory. The pure spinors in ten dimensions are dimensionally reduced and yield the harmonic coordinates. Two anticommuting BRST charges implement Grassmann analyticity and harmonic analyticity. The string field theory action produces the action and field equations for $N=3$ SYM theory in harmonic superspace.

Stony Brook, 2/24/2004

¹ pgrassi@insti.physics.sunysb.edu

² vannieu@insti.physics.sunysb.edu

1. Introduction

Pure spinors [1] in ten dimensions are complex commuting chiral spinorial ghosts $\lambda^{\hat{\alpha}}$ with $\hat{\alpha} = 1, \dots, 16$ satisfying the ten nonlinear constraints

$$\lambda^{\hat{\alpha}} \gamma_{\hat{\alpha}\hat{\beta}}^{\hat{m}} \lambda^{\hat{\beta}} = 0, \quad (1.1)$$

(hats denote 10-dimensional indices). They form the starting point for a new approach to the quantization of the superstring with coordinates $x^{\hat{m}}, \theta^{\hat{\alpha}}$ and $\lambda^{\hat{\alpha}}$ [2]. Due to these constraints on λ , the troublesome second class constraints of the superstring become effectively first class. One can relax these constraints and obtain a covariant formulation by introducing more ghosts as Lagrange multipliers [3]. The result is an $N = 2$ WZNW model [4]. The pure spinors in this covariant approach are real and the BRST charge maps $\theta^{\hat{\alpha}}$ into $\lambda^{\hat{\alpha}}$. In this letter, though, we use complex constrained $\lambda^{\hat{\alpha}}$. Pure spinors also exist in other dimensions [1].

Harmonic superspace (see [5] for a complete review of the subject and references³) was constructed to circumvent the no-go theorems for a full-fledged superspace description of N-extended supersymmetries (susy). The main idea is to let the R -symmetry group $U(N)$ (or $SU(N)$ for $N=4$), which acts on the susy generators, become part of a coset approach. The generators of $U(N)$ are divided into coset generators with coset coordinates u called harmonic variables, and subgroup generators. Superfields depend not only on x^m and half of the $\theta_I^\alpha, \bar{\theta}^{\dot{\alpha}I}$ (with $\alpha, \dot{\alpha} = 1, 2$ and $I = 1, \dots, N$) but also on u 's. For $N = 2, 3, 4$ the cosets most often used are

$$\frac{SU(2)}{U(1)}, \quad \frac{SU(3)}{U(1) \times U(1)}, \quad \frac{SU(4)}{S[U(2) \times U(2)]}, \quad (1.2)$$

respectively, although other choices are also possible [6].

In this letter we present a derivation of four-dimensional harmonic superspaces from ten-dimensional pure spinors by using ordinary dimensional reduction in which we set the extra six coordinates to zero by hand. The spinors $\lambda^{\hat{\alpha}}$ decompose into λ_I^α and $\bar{\lambda}^{\dot{\alpha}I}$ where $I = 1, \dots, 4$ is an $SU(4) \sim SO(6)$ index. The main idea is to factorize the pure spinors

³ Two useful accounts of the subject can be found in [6] and in [7]. Projective harmonic superspace has been introduced in [8]. The application to the AdS/CFT correspondence is studied in [9], and some developments of $N = 4$ harmonic superspace for SYM can be found in [10] and in [11].

$\lambda^{\hat{\alpha}}$ into auxiliary variables λ_a^α and $\bar{\lambda}_a^{\hat{\alpha}}$ with $a = 1, 2$, and harmonic variables u_I^a and \bar{v}^{aI} . In this way we factorize the Lorentz group and the internal symmetry group $SU(4)$. Using this factorization, the pure spinor constraints turn into constraints on λ_a^α and $\bar{\lambda}_a^{\hat{\alpha}}$, and on u_I^a and \bar{v}^{aI} .

Contracting the operator $d_{z\hat{\alpha}}$ in the BRST charge [2]

$$Q = \oint dz \lambda^{\hat{\alpha}} d_{z\hat{\alpha}}, \quad (1.3)$$

with the harmonic coordinates leads to eight spinorial covariant derivatives

$$d_\alpha^a = u_I^a d_\alpha^I, \quad \bar{d}_{\hat{\alpha}}^a = \bar{v}^{aI} \bar{d}_{\hat{\alpha}I}, \quad (1.4)$$

which satisfy the constraints

$$\{d_\alpha^a, d_\beta^b\} = \epsilon_{\alpha\beta} \{\bar{d}_{\hat{\alpha}}^a, \bar{d}_{\hat{\beta}}^b\}, \quad \{d_\alpha^a, \bar{d}_{\hat{\beta}}^b\} = 0, \quad (1.5)$$

as a consequence of the constraints on u and \bar{v} , and in terms of which G(Grassman)-analyticity (dependence on half the θ 's) of superfields is defined.

If one does not provide the information that d_α^a and $\bar{d}_{\hat{\alpha}}^a$ are linear in u_I^a and \bar{v}^{aI} , one loses information. We therefore construct a second BRST charge which only anticommutes with Q_H if d_α^a and $\bar{d}_{\hat{\alpha}}^a$ are factorized as in (1.4). It is constructed from the generators of $U(N)$ represented by the following differential operators⁴

$$d^a_{\alpha'} = u_I^a \partial_{u_{\alpha'}^I} - \bar{u}_{\alpha'}^I \partial_{\bar{u}_a^I}. \quad (1.6)$$

Requiring that the vertex operators are annihilated by these BRST charges should yield the field equations of N=4 harmonic superspace. In this letter we work out the case of N=3 and obtain by truncation the field equations of N=3 SYM theory in harmonic superspace. We end by deducing an action for N=3 SYM theory in harmonic superspace from the Chern-Simons action for string field theory [12].

The present analysis might provide a link between string theory with pure spinors and recent developments in twistor theory [13]. Another interesting aspect not covered in the present letter is deformed harmonic superspace [14]. It would be interesting to discover which kind of harmonic superspace one obtains for suitable Ramond-Ramond background fields [15].

In a future article we intend to extend these results to the N=4 case and construct an action for N=4 SYM theory [16]. In particular, this should give a conceptually simple derivation of the rather complicated measure.⁵

⁴ The R-symmetry group $SU(4)$ corresponds to the Lorentz generators in the extra dimensions. This suggests that the second BRST charge might be obtained by dimensional reduction of the BRST charge in ten dimensions, extended to include the ten dimensional Lorentz generators.

⁵ A similar analysis is pursued in [17].

2. The coordinates of N=4, N=3, and N=2 harmonic superspace from pure spinors

We substitute the decomposition $\lambda^{\dot{\alpha}} = (\lambda_I^\alpha, \bar{\lambda}^{\dot{\alpha}I})$ into the pure spinor constraints, and use the representation of the matrices $\gamma_{\dot{\alpha}\hat{\beta}}^{\hat{m}}$ given in [18]. In this representation the Dirac matrices with $m = 0, 1, 2, 3$ are labelled by $\gamma^{\alpha\dot{\beta}}$ and those for $m = 4, \dots, 9$ are labelled by $\gamma^{IJ} = -\gamma^{JI}$, and all matrix elements are expressed in terms of Kronecker delta's and the epsilon symbols $\epsilon^{\alpha\beta}, \epsilon^{\dot{\alpha}\dot{\beta}}$ and ϵ^{IJKL} . The pure spinor constraints decompose then into the following six plus four constraints

$$\lambda_I^\alpha \epsilon_{\alpha\beta} \lambda_J^\beta + \frac{1}{2} \epsilon_{IJKL} \bar{\lambda}^{\dot{\alpha}K} \epsilon_{\dot{\alpha}\dot{\beta}} \bar{\lambda}^{\dot{\beta}L} = 0, \quad \lambda_I^\alpha \bar{\lambda}^{\dot{\alpha}I} = 0. \quad (2.1)$$

The first relation corresponds to $m = 4, \dots, 9$ while the second one corresponds to $m = 0, 1, 2, 3$. To solve these constraints we adopt the following ansatz

$$\lambda_I^\alpha = \lambda_a^\alpha u_I^a, \quad \bar{\lambda}^{\dot{\alpha}J} = \bar{\lambda}_a^{\dot{\alpha}} \bar{v}^{aJ}, \quad (2.2)$$

where $a = 1, 2$. The new variables u_I^a and \bar{v}^{aJ} are complex and commuting. They carry $GL(2, \mathbf{C})$ and $SU(4)$ indices. The spinors $\lambda_a^\alpha, \bar{\lambda}_a^{\dot{\alpha}}$ are also complex and commuting, and carry a representation of $SL(2, \mathbf{C})$ and $GL(2, \mathbf{C})$. In this way, we separate the Lorentz group from the internal symmetry group $SU(4)$.

The decomposition in (2.2) is left invariant by the gauge transformations

$$u_I^a \rightarrow M^a_b u_I^b, \quad \lambda_a^\alpha \rightarrow \lambda_b^\alpha (M^{-1})^b_a, \quad (2.3)$$

$$\bar{v}^{aJ} \rightarrow \bar{M}^a_b \bar{v}^{bJ}, \quad \bar{\lambda}_a^{\dot{\alpha}} \rightarrow \bar{\lambda}_b^{\dot{\alpha}} (\bar{M}^{-1})^b_a,$$

where M and \bar{M} are independent $GL(2, \mathbf{C})$ matrices. The factorization (2.2) plus the gauge invariance (2.3) yields 16 complex parameters. To reduce to the usual 11 independent complex parameters of pure spinors, we further impose the following two covariant constraints

$$u_I^a \bar{v}^{bI} = 0, \quad \lambda_a^\alpha \epsilon_{\alpha\beta} \epsilon^{ab} \lambda_b^\beta + \bar{\lambda}_a^{\dot{\alpha}} \epsilon_{\dot{\alpha}\dot{\beta}} \epsilon^{ab} \bar{\lambda}_b^{\dot{\beta}} = 0. \quad (2.4)$$

The first one imposes four complex conditions, while the second equation is a single invariant complex condition.

The first constraint in (2.4) and the gauge transformations in (2.3) reduce the 16 complex components of u_I^a and \bar{v}^{aI} to 8 real parameters. This is the same number as the

number of independent parameters of the coset $\frac{U(4)}{U(2) \times U(2)} = \frac{SU(4)}{S(U(2) \times U(2))}$ used in [7] (see also [11] and [9]). The restriction of $U(2) \times U(2)$ to the subgroup $S(U(2) \times U(2))$ is due to second constraint of (2.4). The latter is preserved by the transformations M and \bar{M} only after the identification $\det M = \det \bar{M}$.

To identify the $SU(4)$ of the coset space, we introduce new coordinates $u_I^{a,\dot{b}} = (u_I^{a,1}, u_I^{a,2})$ where

$$u_I^{a,1} = u_I^a, \quad u_I^{a,2} = \epsilon^{ab} v_{bI}, \quad (2.5)$$

and $v_{bI} = (\bar{v}^{bI})^*$. The matrix $u_I^{(a,\dot{b})}$ is a $U(4)$ matrix because the harmonic variables u_I^a and \bar{v}^{aI} satisfy the constraints (2.4) and they can be normalized as follows, using the gauge transformations (2.3),

$$u_I^a \bar{u}_b^I = \delta_b^a, \quad \bar{v}^{aI} v_{bI} = \delta_b^a, \quad (2.6)$$

where $\bar{u}_b^I = (u_I^b)^*$.

To restrict $U(4)$ to $SU(4)$ we choose the gauge ⁶

$$u_I^a \epsilon_{ab} u_J^b - \frac{1}{2} \epsilon_{IJKL} \bar{v}^{aK} \epsilon_{ab} \bar{v}^{bL} = 0. \quad (2.7)$$

This gauge choice is preserved by $S(U(2) \times U(2))$.

The normalizations (2.6) fix 4 real parameters for each $GL(2, \mathbf{C})$ in (2.3). The remaining 7 real parameters of $GL(2, \mathbf{C})$ (remaining after the identification $\det M = \det \bar{M}$), reproduce the subgroup $S(U(2) \times U(2))$. All equations are covariant under this subgroup. Thus the coordinates $u_I^A \equiv u_I^{a,\dot{a}}$, with $A = 1, \dots, 4$, parametrize the coset $\frac{SU(4)}{S(U(2) \times U(2))}$.

Let us turn to N=3 harmonic superspace. If we decompose the λ_I^α 's and the $\bar{\lambda}^{\dot{\alpha}I}$'s into N=3 vectors and N=3 scalars we have $\lambda_I^\alpha = (\lambda_i^\alpha, \psi^\alpha)$ and $\bar{\lambda}^{\dot{\alpha}I} = (\bar{\lambda}^{\dot{\alpha}i}, \bar{\psi}^{\dot{\alpha}})$. In that basis, the pure spinor constraints in (2.1) become

$$\begin{aligned} \lambda_i^\alpha \epsilon_{\alpha\beta} \lambda_j^\beta + \epsilon_{ijk} \bar{\lambda}^{\dot{\alpha}k} \epsilon_{\dot{\alpha}\dot{\beta}} \bar{\psi}^{\dot{\beta}} &= 0, \\ \lambda_i^\alpha \epsilon_{\alpha\beta} \psi^\beta + \epsilon_{ijk} \bar{\lambda}^{\dot{\alpha}j} \epsilon_{\dot{\alpha}\dot{\beta}} \bar{\lambda}^{\dot{\beta}k} &= 0, \\ \lambda_i^\alpha \bar{\lambda}^{\dot{\alpha}i} + \psi^\alpha \bar{\psi}^{\dot{\alpha}} &= 0. \end{aligned} \quad (2.8)$$

The reduction to the N=3 case is obtained by setting $\psi^\alpha = \bar{\psi}^{\dot{\alpha}} = 0$. Inserting this ansatz into the first two equations of (2.8), we obtain

$$\lambda_i^\alpha \epsilon_{\alpha\beta} \lambda_j^\beta = 0, \quad \bar{\lambda}^{\dot{\alpha}j} \epsilon_{\dot{\alpha}\dot{\beta}} \bar{\lambda}^{\dot{\beta}k} = 0, \quad (2.9)$$

⁶ Denoting this relation by $N_{IJ} = 0$, it is clear that $N_{IJ} \bar{v}^{aJ} = 0$ and $\epsilon^{IJKL} N_{KL} u_J^a = 0$ due to (2.4). This leaves the phase of $\det u_I^{a\dot{b}}$ undetermined. The gauge in (2.5) sets this phase to zero.

which is equivalent to requiring that all determinants of order 2 of the matrices λ_i^α and $\bar{\lambda}^{\dot{\alpha}i}$ vanish.⁷ This means that the pure spinors can be factorized into

$$\lambda_i^\alpha = \lambda^\alpha u_i, \quad \bar{\lambda}^{\dot{\alpha}i} = \bar{\lambda}^{\dot{\alpha}} \bar{v}^i \quad (2.10)$$

and the equations (2.8) are solved by

$$\psi^\alpha = \bar{\psi}^{\dot{\alpha}} = 0, \quad u_i \bar{v}^i = 0. \quad (2.11)$$

So for the N=3 case no constraint is needed for λ^α and $\bar{\lambda}^{\dot{\alpha}}$. Notice that the two complex vectors u_i and \bar{v}^i are defined up to a gauge transformation

$$u_i \rightarrow \rho u_i, \quad \lambda^\alpha \rightarrow \rho^{-1} \lambda^\alpha, \quad (2.12)$$

$$\bar{v}^i \rightarrow \sigma \bar{v}^i, \quad \bar{\lambda}^{\dot{\alpha}} \rightarrow \sigma^{-1} \bar{\lambda}^{\dot{\alpha}}$$

where $\rho, \sigma \in \mathbf{C}$. The two real parameters $|\rho|$ and $|\sigma|$ are used to impose the normalizations $u_i \bar{u}^i = 1$ and $v_i \bar{v}^i = 1$. If one also gauges away the overall phases of u_i and \bar{v}^i , the space of harmonic coordinates u_i and \bar{v}^i is parametrized by six real parameters. This coincides with the number of free parameters of the coset $SU(3)/U(1) \times U(1)$. Indeed, we can construct 3×3 matrices $(u_i^1, u_i^2, u_i^3) = (u_i^{(1,0)}, u_i^{(0,-1)}, u_i^{(-1,1)})$ as follows

$$u_i^1 \equiv u_i^{(1,0)} = u_i, \quad u_i^2 \equiv u_i^{(-1,1)} = \epsilon_{ijk} \bar{v}^j \bar{u}^k, \quad u_i^3 \equiv u_i^{(0,-1)} = v_i. \quad (2.13)$$

where $\bar{u}^i = (u_i)^*$ and $v_i = (\bar{v}^i)^*$. Fixing the phases of u_i^1 and u_i^3 , the u_i^I form $SU(3)$ matrices which are coset representatives of $\frac{SU(3)}{U(1) \times U(1)}$. The $U(1) \times U(1)$ transformations generate the phases $\arg(\rho)$ and $\arg(\sigma)$. The notation $u_i^{(a,b)}$ indicates the $U(1) \times U(1)$ charges of the harmonic variables and they satisfy the hermiticity property $\overline{u_i^{(a,b)}} = u^{i(-a,-b)}$. We denote by u_I^i the inverse harmonics

$$u_I^i u_i^J = \delta_I^J, \quad u_I^I u_I^j = \delta_i^j, \quad \det u = \epsilon^{ijk} u_i^1 u_j^2 u_k^3 = 1. \quad (2.14)$$

For later use we also list the components of the inverse matrix u_I^i :

$$u_1^i \equiv u^{i(-1,0)} = \overline{u_i^{(1,0)}} = \bar{u}^i, \quad u_2^i \equiv u^{i(1,-1)} = \epsilon^{ijk} v_j u_k, \quad u_3^i \equiv u^{i(0,1)} = \bar{v}^i. \quad (2.15)$$

⁷ It is well-known (and easy to check) that if two of the 2×2 submatrices have vanishing determinant, so does the third. This implies (2.10).

Finally, we consider a further reduction to N=2. We decompose the N=3 pure spinors λ_i^α and $\bar{\lambda}^{\dot{\alpha}i}$ into a vector of N=2 and a singlet, $\lambda_i^\alpha = (\lambda_{\mathcal{I}}^\alpha, \lambda_3^\alpha)$ and $\bar{\lambda}^{\dot{\alpha}i} = (\bar{\lambda}^{\dot{\alpha}\mathcal{I}}, \bar{\lambda}^{\dot{\alpha}3})$ where $\mathcal{I} = 1, 2$. We set λ_3^α and $\bar{\lambda}_3^{\dot{\alpha}}$ to zero. The pure spinor equations (2.8) reduce then to

$$\lambda_{\mathcal{I}}^\alpha \epsilon_{\alpha\beta} \lambda_{\mathcal{J}}^\beta \epsilon^{\mathcal{I}\mathcal{J}} = 0, \quad \bar{\lambda}^{\dot{\alpha}\mathcal{J}} \epsilon_{\dot{\alpha}\dot{\beta}} \bar{\lambda}^{\dot{\beta}\mathcal{K}} \epsilon_{\mathcal{J}\mathcal{K}} = 0, \quad \lambda_{\mathcal{I}}^\alpha \bar{\lambda}^{\dot{\alpha}\mathcal{I}} = 0. \quad (2.16)$$

The first two equations imply that $\lambda_{\mathcal{I}}^\alpha$ and $\bar{\lambda}^{\dot{\alpha}\mathcal{I}}$ are factorized into $\lambda_{\mathcal{I}}^\alpha = \lambda^\alpha u_{\mathcal{I}}$ and $\bar{\lambda}^{\dot{\alpha}\mathcal{I}} = \bar{\lambda}^{\dot{\alpha}} \bar{v}^{\mathcal{I}}$ where $u_{\mathcal{I}} \bar{v}^{\mathcal{I}} = 0$. The vector $\bar{v}^{\mathcal{I}}$ is proportional to $\epsilon^{\mathcal{I}\mathcal{J}} u_{\mathcal{J}}$. Hence without loss of generality one may write

$$\lambda_{\mathcal{I}}^\alpha = \lambda^\alpha u_{\mathcal{I}}, \quad \bar{\lambda}^{\dot{\alpha}\mathcal{I}} = \bar{\lambda}^{\dot{\alpha}} \epsilon^{\mathcal{I}\mathcal{J}} u_{\mathcal{J}}. \quad (2.17)$$

With this parametrization of the N=2 case there are neither constraints on the λ 's nor on the u 's.

The vector $u_{\mathcal{I}}$ yields the usual parametrization of N=2 harmonic superspace [5]. Namely, one introduces the $SU(2)$ matrix $(u_{\mathcal{I}}^+, u_{\mathcal{I}}^-)$ where $u_{\mathcal{I}}^+ = u_{\mathcal{I}}$ and $u_{\mathcal{I}}^- = (u^{+\mathcal{I}})^*$ with $u_{\mathcal{J}}^+ = \epsilon_{\mathcal{J}\mathcal{K}} u^{+\mathcal{K}}$. The coset $SU(2)/U(1)$ is obtained by dividing by the subgroup $U(1)$ which generates the phases $u_{\mathcal{I}}^\pm \rightarrow e^{\pm i\alpha} u_{\mathcal{I}}^\pm$. In fact, eqs. (2.17) are defined up to a rescaling of $\lambda^\alpha, \bar{\lambda}^{\dot{\alpha}}$ and of $u_{\mathcal{I}}$ given by $u_{\mathcal{I}} \rightarrow \rho u_{\mathcal{I}}$, for $\rho \neq 0$. This yields the compact space \mathbf{CP}^1 .

3. N=3 Harmonic Superspace for SYM Theory from Superstrings

The field equation for $D = 4, N = 3$ SYM-theory in ordinary (not harmonic) superspace are given by [19]

$$\{\nabla_\alpha^i, \nabla_\beta^j\} = \epsilon_{\alpha\beta} \bar{W}^{ij}, \quad \{\bar{\nabla}_{\dot{\alpha}i}, \bar{\nabla}_{\dot{\beta}j}\} = \epsilon_{\dot{\alpha}\dot{\beta}} W_{ij}, \quad (3.1)$$

$$\{\nabla_\alpha^i, \bar{\nabla}_{\dot{\beta}j}\} = \delta_j^i \nabla_{\alpha\dot{\beta}}.$$

The coordinates for this N=3 superspace, $(x^m, \theta_i^\alpha, \bar{\theta}^{\dot{\alpha}i})$, are obtained by imposing the constraint $\theta_4^\alpha = \bar{\theta}^{\dot{\alpha}4} = 0$. Since θ 's transform into λ 's under BRST transformations we also impose for consistency $\lambda_4^\alpha = \bar{\lambda}^{\dot{\alpha}4} = 0$.

Using the decomposition of the N=3 spinors λ_i^α and $\bar{\lambda}^{\dot{\alpha}i}$ given in (2.10), and contracting the harmonic variables with the operators $d_{z\hat{\alpha}}$ in (1.3) yields two new spinorial operators

$$Q_G = \lambda^\alpha d_\alpha^1 + \bar{\lambda}^{\dot{\alpha}} \bar{d}_{3\dot{\alpha}}.$$

$$d_\alpha^1 = u_i d_\alpha^i = u_i^1 d_\alpha^i = u_i^{(1,0)} d_\alpha^i, \quad \bar{d}_{3\dot{\alpha}} = \bar{v}^i \bar{d}_{\dot{\alpha}i} = u_3^i \bar{d}_{\dot{\alpha}i} = u^{i(0,1)} \bar{d}_{\dot{\alpha}i}. \quad (3.2)$$

The operator d_α^1 corresponds to $\xi_i D_\alpha^i$ and $\bar{d}_{3\dot{\alpha}}$ to $\eta^i \bar{D}_{\dot{\alpha}i}$ in [5].

Due to the constraints on the u 's the operators d_α^1 and $\bar{d}_{3\dot{\alpha}}$ satisfy the commutation relations

$$\{d_\alpha^1, d_\beta^1\} = 0, \quad \{d_\alpha^1, \bar{d}_{3\dot{\beta}}\} = 0, \quad \{\bar{d}_{3\dot{\alpha}}, \bar{d}_{3\dot{\beta}}\} = 0. \quad (3.3)$$

To derive these relations one may use the dimensionally reduced relations $\{d_\alpha^i, d_\beta^j\} = \epsilon_{\alpha\beta} \Pi^{ij}$, $\{\bar{d}_{\dot{\alpha}i}, \bar{d}_{\dot{\beta}j}\} = \epsilon_{\dot{\alpha}\dot{\beta}} \bar{\Pi}_{ij}$ and $\{d_\alpha^i, \bar{d}_{\dot{\alpha}j}\} = \delta_j^i \Pi_{\alpha\dot{\beta}}$. Hence Q_G (where G stands for Grassmann) is nilpotent for any λ^α and $\bar{\lambda}^{\dot{\alpha}}$.

The BRST operator Q_G implements naturally the G-analyticity on the space of superfields $\Phi(x, \theta, \bar{\theta}, \lambda, \bar{\lambda}, u)$. A superfield with ghost number zero is given by $\Phi(x, \theta, \bar{\theta}, u)$ and G-analyticity means $Q_G \Phi = 0$ which implies $D_\alpha^1 \Phi = \bar{D}_{3\dot{\alpha}} \Phi = 0$ (since $\{d_\alpha^1, \Phi(x, \theta, \bar{\theta}, \lambda, \bar{\lambda}, u)\} = D_\alpha^1 \Phi(x, \theta, \bar{\theta}, \lambda, \bar{\lambda}, u)$ and similarly for $\bar{d}_{3\dot{\alpha}}$). Such a superfield is called a G-analytic superfield in [5]. A generic superfield $\Phi(x, \theta, \bar{\theta}, \lambda, \bar{\lambda}, u)$ with ghost number one can be parametrized in terms of two u -dependent spinorial superfields $A_\alpha, \bar{A}_{\dot{\alpha}}$ as follows

$$\Phi^{(1)}(x, \theta, \bar{\theta}, \lambda, \bar{\lambda}, u) = \lambda^\alpha A_\alpha + \bar{\lambda}^{\dot{\alpha}} \bar{A}_{\dot{\alpha}}, \quad (3.4)$$

and $\{Q_G, \Phi^{(1)}\} = 0$ implies the following constraints on these superfields

$$D_\alpha^1 A_\beta + D_\beta^1 A_\alpha = 0, \quad \bar{D}_{3\dot{\alpha}} \bar{A}_{\dot{\beta}} + \bar{D}_{3\dot{\beta}} \bar{A}_{\dot{\alpha}} = 0, \quad D_\alpha^1 \bar{A}_{\dot{\beta}} + \bar{D}_{3\dot{\beta}} A_\alpha = 0. \quad (3.5)$$

Assuming that A_α and $A_{\dot{\alpha}}$ factorize in the same way as $D_\alpha^1 = u_i D_\alpha^i$ and $\bar{D}_{3\dot{\alpha}} = \bar{v}^i \bar{D}_{\dot{\alpha}i}$, so $A_\alpha = u_i A_\alpha^i$ and $A_{\dot{\alpha}} = \bar{v}^i A_{\dot{\alpha}i}$, the equations (3.5) reproduce (3.1). We stress that (3.5), unlike (3.1), do not put the theory on-shell; only the extra assumption of the factorization of A_α and $A_{\dot{\alpha}}$ puts the theory on-shell.

Gauge transformations are generated by a ghost-number zero scalar superfield $\Omega^{(0)}$. To lowest order in $\Phi^{(1)}$ they read $\delta\Phi^{(1)} = \{Q_G, \Omega^{(0)}\}$ which yields $\delta A_\alpha = D_\alpha \Omega$ and $\delta A_{\dot{\alpha}} = \bar{D}_{\dot{\alpha}} \Omega$. Equations (3.5) are easily solved in D=4; they imply that the superfields A_α and $\bar{A}_{\dot{\alpha}}$ are pure gauge. Hence the Q_G -cohomology in the space of superfields with ghost number 1 vanishes.

To determine on which harmonic variables superfields depend, we construct a second BRST operator Q_H which is constructed from the $SU(3)$ generators

$$d^a{}_b = u_i^a \partial_{u_i^b} - u_b^i \partial_{u_i^a} = u_i^a p_b^i - u_b^i p_i^a. \quad (3.6)$$

where p_b^i can be represented by $\partial/\partial u_i^b$ and similarly for p_i^b . These generators split into three raising operators $d_2^1 = d^{(2,-1)}$, $d_3^2 = d^{(-1,2)}$, $d_3^1 = d^{(1,1)}$, three lowering operators $d_1^2 = d^{(-2,1)}$, $d_2^3 = d^{(1,-2)}$, $d_1^3 = d^{(-1,-1)}$, and two Cartan generators d_1^1 and d_2^2 . The raising operators commute with Q_G

$$[d^{(2,-1)}, d_\alpha^1] = [d^{(-1,2)}, d_\alpha^1] = [d^{(1,1)}, d_\alpha^1] = 0, \quad (3.7)$$

$$[d^{(2,-1)}, d_{3\dot{\alpha}}] = [d^{(-1,2)}, d_{3\dot{\alpha}}] = [d^{(1,1)}, d_{3\dot{\alpha}}] = 0.$$

and form an algebra, in particular $[d^{(2,-1)}, d^{(-1,2)}] = d^{(1,1)}$. This suggests to construct a new nilpotent BRST operator Q_H

$$Q_H = \xi_1^3 d_3^1 + \xi_1^2 d_2^1 + \xi_2^3 d_3^2 - \beta_3^1 \xi_1^2 \xi_2^3, \quad (3.8)$$

where we introduced new pairs of anticommuting (anti)ghosts (ξ_1^3, β_3^1) , (ξ_1^2, β_2^1) , (ξ_2^3, β_3^2) with canonical anticommutation relations. It is convenient to use a notation in which the $U(1) \times U(1)$ weights are made explicit $\xi_1^3 \equiv \xi^{(-1,-1)}$, $\xi_1^2 \equiv \xi^{(-2,1)}$ and $\xi_2^3 = \xi^{(1,-2)}$.

Since Q_H and Q_G anticommute their sum Q_{tot} is obviously nilpotent. A generic superfield $\Phi^{(1)}$ with ghost number one can be decomposed into the following pieces

$$\Phi^{(1)} = \lambda^\alpha A_\alpha^{(1,0)} + \bar{\lambda}^{\dot{\alpha}} \bar{A}_{\dot{\alpha}}^{(0,1)} + \xi_1^3 A^{(1,1)} + \xi_1^2 A^{(2,-1)} + \xi_2^3 A^{(-1,2)} \quad (3.9)$$

where $A_\alpha^{(1,0)}$, $\bar{A}_{\dot{\alpha}}^{(0,1)}$, $A^{(2,-1)}$, $A^{(-1,2)}$ and $A^{(1,1)}$ are harmonic superfields (superfields which depend on the variables u). The harmonic weights of the superfields follow from requiring that $\Phi^{(1)}$ has zero harmonic weight, just like the BRST charge Q_{tot} . Note that $\Phi^{(1)}$ depends only upon the variables $x, \theta, \bar{\theta}, \lambda, \bar{\lambda}$'s and u 's and not upon the conjugated momenta as a consequence of quantum mechanical rules. This forbids ghost-number one combinations of the form $\beta\xi\xi, \beta\xi\lambda, \dots$

The equations of motion for N=3 SYM follow from the BRST-cohomology equations

$$\{Q_{tot}, \Phi^{(1)}\} + \frac{1}{2}\{\Phi^{(1)}, \Phi^{(1)}\} = 0. \quad (3.10)$$

Decomposing the superfield $\Phi^{(1)}$ into $\Phi_H^{(1)} + \Phi_G^{(1)}$, where $\Phi_H^{(1)}$ denotes the terms with ξ -ghosts and $\Phi_G^{(1)}$ the terms with λ -ghosts, the Maurer-Cartan equations in (3.10) decompose as follows

$$\{Q_G, \Phi_G^{(1)}\} + \frac{1}{2}\{\Phi_G^{(1)}, \Phi_G^{(1)}\} = 0, \quad (3.11)$$

$$\{Q_G, \Phi_H^{(1)}\} + \{Q_H, \Phi_G^{(1)}\} + \{\Phi_G^{(1)}, \Phi_H^{(1)}\} = 0, \quad (3.12)$$

$$\{Q_H, \Phi_H^{(1)}\} + \frac{1}{2}\{\Phi_H^{(1)}, \Phi_H^{(1)}\} = 0. \quad (3.13)$$

This system of equations is invariant under the infinitesimal gauge transformation

$$\Phi^{(1)} \rightarrow \Phi^{(1)} + \{Q_{tot}, \Omega\} + \{\Phi^{(1)}, \Omega\}, \quad (3.14)$$

where Ω is a generic harmonic superfield with ghost number zero. According to the above decomposition of $\Phi^{(1)}$, one obtains $\delta\Phi_G^{(1)} = \{Q_G, \Omega\} + \{\Phi_G^{(1)}, \Omega\}$ and $\delta\Phi_H^{(1)} = \{Q_H, \Omega\} + \{\Phi_H^{(1)}, \Omega\}$.

To reduce the system of equations in (3.11)-(3.13) to the field equations of harmonic superspace, we use the fact that Q_G has no cohomology. This implies that equation (3.11) is solved by a pure gauge superfield $\Phi_G^{(1)} = e^{-i\Delta}(Q_G e^{i\Delta})$ where Δ is a ghost-number zero superfield known in the literature as the *bridge* (see for example [5]). Also the BRST cohomology of Q_H vanishes on the unconstrained superspace and therefore one can also solve the system (3.11)-(3.13) starting from the last equation.

In the harmonic superspace framework, one usually employs the bridge superfield $\Delta(x, \theta, \bar{\theta}, u)$ to bring the spinorial covariant derivatives to the ‘pure gauge’ form

$$\nabla_\alpha^{(1,0)} = e^{-i\Delta} d_\alpha^{(1,0)} e^{i\Delta}, \quad \bar{\nabla}_{\dot{\alpha}}^{(0,1)} = e^{-i\Delta} \bar{d}_{\dot{\alpha}}^{(0,1)} e^{i\Delta}. \quad (3.15)$$

Here the bridge is seen as the most general solution of (3.11). By making a finite gauge transformation which sets $\Phi_G^{(1)} = 0$, the gauge transformed $\Phi_H^{(1)}$ is given by

$$e^{-i\Delta}(\Phi_H^{(1)} + Q_H)e^{i\Delta} = \xi_1^3 V^{(1,1)} + \xi_1^2 V^{(2,-1)} + \xi_2^3 V^{(-1,2)}. \quad (3.16)$$

Equation (3.12) becomes

$$D_\alpha^{(1,0)} V^{(2,-1)} = D_\alpha^{(1,0)} V^{(-1,2)} = D_\alpha^{(1,0)} V^{(1,1)} = 0, \quad (3.17)$$

$$\bar{D}_{\dot{\alpha}}^{(0,1)} V^{(2,-1)} = \bar{D}_{\dot{\alpha}}^{(0,1)} V^{(-1,2)} = \bar{D}_{\dot{\alpha}}^{(0,1)} V^{(1,1)} = 0,$$

expressing the G -analyticity of the harmonic connections $V^{(1,1)}$, $V^{(2,-1)}$ and $V^{(-1,2)}$. The last equation (3.13) finally gives the SYM equations of motion of N=3 harmonic superspace

$$D^{(2,-1)} V^{(-1,2)} - D^{(-1,2)} V^{(2,-1)} + [V^{(2,-1)}, V^{(-1,2)}] = V^{(1,1)},$$

$$\begin{aligned}
D^{(2,-1)}V^{(1,1)} - D^{(1,1)}V^{(2,-1)} + [V^{(2,-1)}, V^{(1,1)}] &= 0, \\
D^{(-1,2)}V^{(1,1)} - D^{(1,1)}V^{(-1,2)} + [V^{(-1,2)}, V^{(1,1)}] &= 0.
\end{aligned} \tag{3.18}$$

where the harmonic derivatives $D^{(1,1)}, D^{(2,-1)}$ and $D^{(-1,2)}$ represent the action of $d^{(1,1)}, d^{(2,-1)}$ and $d^{(-1,2)}$ on u -dependent superfields. These are the field equations of $N = 3$ SYM harmonic superspace, see eq. (12.57) in [5]. Equations (3.17)-(3.18) are invariant under the gauge transformations

$$\begin{aligned}
\delta V^{(2,-1)} &= D^{(2,-1)}\omega + [V^{(2,-1)}, \omega], & \delta V^{(-1,2)} &= D^{(-1,2)}\omega + [V^{(-1,2)}, \omega], \\
\delta V^{(1,1)} &= D^{(1,1)}\omega + [V^{(1,1)}, \omega],
\end{aligned} \tag{3.19}$$

where the superfield ω satisfies

$$D_\alpha^{(1,0)}\omega = 0, \quad \bar{D}_{\dot{\alpha}}^{(0,1)}\omega = 0. \tag{3.20}$$

4. The Action and Measure for $N = 3$ SYM theory

We start from the observation that the field equations (3.10) are of Chern-Simons form and can be derived from an action of the form

$$S_{CS} = \int d\mu \left(\Phi^{(1)} Q_{tot} \Phi^{(1)} + \frac{2}{3} \Phi^{(1)} \star \Phi^{(1)} \star \Phi^{(1)} \right) \tag{4.1}$$

where \star denotes conventional matrix multiplication. The measure $d\mu$ has to be determined.

Instead of dimensionally reducing (4.1) we follow a different path. We have to define the integration measure for all zero modes in the theory. Since we are dealing with worldline models, the only contribution comes from the zero modes of $x^\mu, \theta_i^\alpha, \bar{\theta}^{\dot{\alpha}i}, \lambda_i^\alpha, \bar{\lambda}^{\dot{\alpha}i}, u_i^I$ and $\xi_3^1, \xi_1^2, \xi_2^1$. The set of ghosts $\lambda_i^\alpha, \bar{\lambda}^{\dot{\alpha}i}$ pertains to the BRST charge Q_G which implements the G -analyticity. Therefore, they implement kinematical constraints on the theory expressed by the equations:

$$[Q_G, S_{N=3}] = 0, \quad [Q_G, d\mu_H] = 0, \tag{4.2}$$

where $S_{N=3}$ is the off-shell $N = 3$ action and $d\mu_H$ is the invariant measure in the space of the zero modes of $x^\mu, \theta_i^\alpha, \bar{\theta}^{\dot{\alpha}i}, u_i^I$ and $\xi_3^1, \xi_1^2, \xi_2^1$. In addition, $S_{N=3}$ has zero ghost number, while $d\mu_H$ has ghost number three. Form [2] and [20] it is known that $d\mu_H \in H^3(Q_H)$. This implies that $d\mu_H = d\xi_3^1 d\xi_1^2 d\xi_2^1 d\mu'$ where the measure $d\mu' = d\mu'(x^\mu, \theta_i^\alpha, \bar{\theta}^{\dot{\alpha}i}, u_i^I)$ has to be fixed by the G -analyticity (4.2).

First we consider the space formed by $x^\mu, \theta_i^\alpha, \bar{\theta}^{\dot{\alpha}i}$. The conditions in (4.2) select the analytic subspace $(x_A^m, \theta_\alpha^{(0,1)}, \theta_\alpha^{(1,-1)}, \bar{\theta}_{\dot{\alpha}}^{(1,0)}, \bar{\theta}_{\dot{\alpha}}^{(-1,1)})$ where $\theta^{(a,b)} = u^{i(a,b)}\theta_i$, and $x_A^{\alpha\dot{\alpha}} = x^{\alpha\dot{\alpha}} + 2i\theta^{\alpha(-1,0)}\bar{\theta}^{\dot{\alpha}(1,0)} + 2i\theta^{\alpha(0,1)}\bar{\theta}^{\dot{\alpha}(0,-1)}$. Therefore the only invariant measure is given by

$$d\mu' = d^4x_A d^2\theta^{(0,1)} d^2\theta^{(1,-1)} d^2\bar{\theta}^{(1,0)} d^2\bar{\theta}^{(-1,1)} d\mu_u \quad (4.3)$$

where $d\mu_u$ is the measure for the harmonic variables. In order to derive a Q_G invariant measure $d\mu_u$, we introduce the new variables (projective harmonic variables [21])

$$z_1 = u_1/u_3, \quad z_2 = u_2/u_3, \quad z_3 = v_1/v_2. \quad (4.4)$$

The three raising and three lowering operators are three Lie derivatives whose duals are six one-forms whose product gives the integration measure on $SU(3)/U(1) \times U(1)$. This is the Haar measure for $SU(3)/U(1) \times U(1)$ given by [21]

$$d\mu_u = \frac{\prod_{i=1}^3 dz_i d\bar{z}_i}{(1 + |z_1|^2 + |z_2|^2)^4 (1 + |z_3|^2 + |z_2 + z_1 z_3|^2)^2}. \quad (4.5)$$

5. Acknowledgemnets

We thank N. Berkovits, M. Porrati, G. Policastro, M. Roček and W. Siegel for useful discussions. This work was partly funded by NSF Grants PHY-0098527. PAG thanks L. Castellani and A. Lerda for discussions and financial support.

References

- [1] É. Cartan, *Lecons sur la théorie des spineurs*, Hermann, Paris (1937); C. Chevalley, *The algebraic theory of Spinors*, Columbia Univ. Press., New York, 1954; R. Penrose and W. Rindler, *Spinors and Space-Time*, Cambridge Univ. Press, Cambridge (1984); P. Furlan and R. Raczka, J. Math. Phys. **26**, 3021 (1985); P. Budinich and A. Trautman, *The spinorial chessboard*, Springer, New York (1989); P.S. Howe, Phys. Lett. B258 (1991) 141, Addendum-ibid.B259 (1991) 51; P.S. Howe, Phys. Lett. B273 (1991) 90.
- [2] N. Berkovits, JHEP 0004, 018 (2000); N. Berkovits, JHEP **0109**, 016 (2001) [hep-th/0105050]. N. Berkovits, Int. J. Mod. Phys. A **16**, 801 (2001); N. Berkovits, *ICTP lectures on covariant quantization of the superstring*, hep-th/0209059.
- [3] P. A. Grassi, G. Policastro, M. Porrati and P. van Nieuwenhuizen, JHEP **10** (2002) 054, [hep-th/0112162]; P. A. Grassi, G. Policastro, and P. van Nieuwenhuizen, JHEP **11** (2002) 004, [hep-th/0202123].
- [4] P. A. Grassi, G. Policastro and P. van Nieuwenhuizen, Nucl. Phys. B **676**, 43 (2004) [hep-th/0307056].
- [5] A. S. Galperin, E. A. Ivanov, V. I. Ogievetsky and E. S. Sokatchev, *Harmonic Superspace*, Cambridge Univ. Press, 2001. A. Galperin, E. Ivanov, S. Kalitzin, V. Ogievetsky and E. Sokatchev, Class. Quant. Grav. **1**, 469 (1984); A. Galperin, E. Ivanov, S. Kalitzin, V. Ogievetsky and Sokatchev, Class. Quant. Grav. **2**, 155 (1985).
- [6] P. S. Howe and G. G. Hartwell, Class. Quant. Grav. **12**, 1823 (1995).
- [7] G. G. Hartwell and P. S. Howe, Int. J. Mod. Phys. A **10**, 3901 (1995) [hep-th/9412147].
- [8] A. Karlhede, U. Lindstrom and M. Rocek, Phys. Lett. B **147**, 297 (1984); S. J. Gates, C. M. Hull and M. Rocek, Nucl. Phys. B **248**, 157 (1984).
- [9] S. Ferrara and E. Sokatchev, Lett. Math. Phys. **52**, 247 (2000) [hep-th/9912168]; L. Andrianopoli, S. Ferrara, E. Sokatchev and B. Zupnik, Adv. Theor. Math. Phys. **3**, 1149 (1999) [hep-th/9912007].
- [10] B. M. Zupnik, [hep-th/0308204].
- [11] B. M. Zupnik, Nucl. Phys. Proc. Suppl. **102**, 278 (2001) [hep-th/0104114].
- [12] E. Witten, Nucl. Phys. B **268**, 253 (1986); E. Witten, [hep-th/9207094].
- [13] E. Witten, hep-th/0312171.
- [14] S. Ferrara and E. Sokatchev, Phys. Lett. B **579**, 226 (2004) [hep-th/0308021]; E. Ivanov, O. Lechtenfeld and B. Zupnik, hep-th/0308012.
- [15] J. de Boer, P. A. Grassi and P. van Nieuwenhuizen, Phys. Lett. B **574**, 98 (2003) [hep-th/0302078]; H. Ooguri and C. Vafa, Adv. Theor. Math. Phys. **7**, 53 (2003) [hep-th/0302109].
- [16] P. A. Grassi, M. Roček, and P. van Nieuwenhuizen, in preparation
- [17] M. Movshev and A. Schwarz, [hep-th/0311132].

- [18] J. Harnad and S. Shnider, Comm. Math. Phys. **106**, 183 (1986)
- [19] M.F. Sohnius, Nucl. Phys. B **136**, 461 (1978); E. Witten, Phys. Lett. B **77**, 394 (1978); E. Witten, Nucl. Phys. B **266**, 245 (1986).
- [20] N. Berkovits, M. T. Hatsuda and W. Siegel, Nucl. Phys. B **371**, 434 (1992) [hep-th/9108021].
- [21] A. A. Roslyi and A. S. Schwarz, Commun. Math. Phys. **105**, 645 (1986).